Figural Representations on Theory of Mathematical Representations: Operational Diagram and Ways to See

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Abstract

In this article, the Theory of Mathematical Representation (Isoda, 1990) was applied to distinguish different types of pictures, drawings, and figures that can be regarded as mathematical representations, alongside algebraic and other forms of representation. To clarify the unique interpretation of written mathematical representation, Isoda defined its nature in terms of context/objective, focused symbol, and focused operation. And to explain 'focus,' the ways to see is explained by using Curriculum Framework to develop mathematical thinking (Isoda, Teh, Gan, 2024)). Using this framework, some figural representations can also be recognized as a form of mathematical representation when used as tools for thinking. This perspective highlights the necessity of teaching figural representations in Japanese origin mathematics textbooks (Isoda & Maitree, 2013), not merely as explanatory tools used by teachers on an ad hoc basis, but as essential thinking tools for students' future learning.

Keywords: Mathematical Thinking, Semiotics, Diagram, Operation, Intuition

1. Introduction

Tall, D. (2013) distinguished three worlds of mathematics by Embodiment, Symbolism and Formalism. Formal representations of mathematics such as the worlds of Symbolism and Formalism are usually expressed through algebraic notation, such as those written using MathType in Microsoft Word. In history of mathematics education, Arithmetic which introduced Europe in Middle Age through such as Fibonacci's Libre Abaci (Stingler, 2002) had used to be represented by Column methods until algebra on the Era of Descartes (Inprasitha, Isoda & Araya, 2023). On Libre Abaci, Arabic numeral from 1 to 9 was introduced as figure and 0 was

called ‘zepher’ instead of alphabet ‘o.’ From this perspective, Arithmetic expressions such as 1+1=2 were formalized as with algebraic notation. On the other hands, figural representations in Embodiment are not included in MathType. Figural representations, which are formally used in school mathematics and taught to students for both drawing and interpretation, include graphs on various coordinate systems (such as number lines and vectors), statistical graphs, and geometric figures. Tables and flowcharts, including arrows, which represent correspondence and relations on two-dimensional diagrams, can also be considered as a kind of formal figural representation. These representations are typically used by students as thinking tools to interpret mathematical meanings of algebraic notation more deeply and efficiently. What about other types of figural representations? Teachers in primary and lower secondary schools often use additional diagrams to explain the meanings of formal representations in school mathematics. Manipulative physical objects which can also be represented by figural representations like pictures. Unfortunately, in so many cases, students are not asked to draw these figures themselves as thinking tools—except in note-taking—because teachers mainly use them as for scaffolding to support learning of formal representations. Figures for scaffolding look unnecessary after students can manage algebraic-formal representations, directly. What kind of figural representations can be seen as a part of mathematical representations as necessary thinking tools.

In mathematics education research, students’ figural drawings that are not formal are referred to as informal representations. It enhances to establish learning trajectory which emergent formal representation from informal representations (Gravemeijer, K., 1999). How we design it in long term development beyond grades. In countries where teachers must follow government-approved textbooks, such as Japan, teachers try to develop students to think by and for themselves by using figural representations which students have previously learned and used, as these are systematically introduced in earlier grades for providing the thinking tools in later grades. In particular, Japan-origin textbooks, such as the Thai edition (Isoda & Maitree, 2013), consistently employ similar figural representations as thinking tools to challenge new problems by adopting what students have already learned.

For long term development, this article answers what kind of figural representations are possible to see as Mathematical Representation as for students thinking tools by Theory of Mathematical Representation (Isoda, 1990). Isoda (1990, 1993, 2015, 2018) established it to generalize Mathematization process by Hans Freudenthal (1972) which reorganize mathematics through objectifying mathematical experience.

2. Theory for Mathematical Representations (Part I)

Before focusing on Figural Representations, firstly we would like to know the nature of Mathematical Representation with the case of Algebraic Representation.

2-1. Objective and Context: Ways to see it as

Mathematical representations convey objectives and contexts by using specific sequential forms, which can be interpreted as mathematical sentences if the forms are understood.

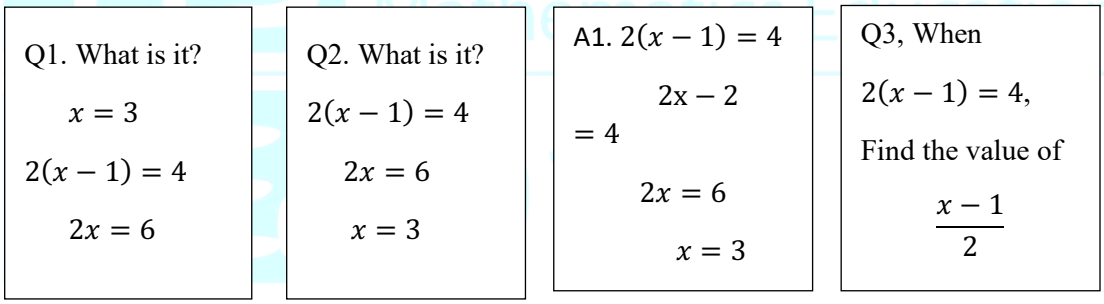


Figure 1. Various ways to see it as.

Figure 1, one possible interpretation is a family of equations where each solution is 3—that is, “a set of equations whose x-value is 3,” or “if we solve each equation, the answer is 3.” If we see the set from this perspective, the set can be interpreted as a family of equations. Otherwise, it is just different equations and cannot interpret meaningfully as a family. In Q2 of Figure 1, the possible interpretation is “the solution of the equation $2(x-1) = 4$” (see Q2 and A1), or, if explained step by step: “from the first line to the second, expand and transpose; from the second to the third, divide both sides by 2” (see A1). In mathematics, the interpretation of Q2 is referred to as ‘the solving an equation’, which is the technical term for this kind of sequential representation of equations. It also specifies both the objective and

the context. On these discussions, if we see only "$x = 3$" as a single form, we cannot determine the objective without adding a sequence of equations. Indeed, "$x = 3$" can indicate different contexts, such as "substitute $x = 3$ into other equations in Q1" or "the solution is $x = 3$" in Q2. In contrast, if an interpretable set of sequential forms by mathematical terms is provided, we can uniquely specify both the objective and the context.

In Q3 of Figure 3, a mathematical problem is presented in a sentence that includes both an equation and an expression. There are at least two possible solution approaches: The first approach for finding the value of $\frac{x-1}{2}$, we first need to know the value of x, so we solve $2(x-1) = 4$ and then substitute the answer to $\frac{x-1}{2}$. The second approach is to notice that the problem contains the same term, $x - 1$, twice. From this, first, find $x - 1 = 2$, and then substitute it directly into $\frac{x-1}{2}$.

As G. Polya (1945) pointed out, a problem can be divided into sub-problems in order to find solutions. The entire solution can be summarized in brief statement, and to explain further each sub-solution can also be summarized in a statement. These statements explain why certain solving steps are taken. Furthermore, each sub-solution can be divided into several steps. Although each step can be described by the terms of its operations, these only indicate the logical correctness in between lines. Even if we accumulate the sequence of all such steps, it provides only logical correctness, but does not necessarily reveal the objective and context—that is, the Gestalt view of why we are performing them in order to solve the problem.

On this hierarchical structure, the whole solution and its sub-solutions can be interpreted in terms of why, while the individual steps can be interpreted in terms of correctness.

2-2. Focused Symbols and Operations

Freudenthal (1983) analyzed the terminology used in school mathematics to clarify mathematical entities, which indicate the objects of thought. The layers, which explained as hierarchical structure, can be analyzed through Peirce's semiotics, which explains how the

meaning of a sign is fixed by the representamen (sign), the semiotic object, and the interpretant (Atkin, 2023; Saengpun and Inprasitha, 2012).

In mathematical problem solving, the representamen can be viewed at several layers such as the whole entire solution as top layer, the sub-solution as second layer, and the individual steps as third layer at least on Q3. The semiotic object for interpretation changes across these layers, and mathematical terminology is used to distinguish these differences.

From the perspective of the objective in solving, the way symbols and operations are treated as objects depends on how we interpret the problem and solutions. For example, in the first solution of Q3, the focused symbol is x, and the focused operations are solving for x using the properties of equations. In the second solution, the focused symbol is $x-1$, and the focused operation is substitution to $x-1$. In mathematical terminology, in this case, the expression $x-1$ is treated as 'a value'—reflecting a way to see the expression and equation as a Gestalt.

2-3. Definition (Part I): Mathematical Representation

A unit of mathematical representation, as a specific set of sequential forms, is defined by its objective, focused symbols, and focused operations. It can be mathematically interpreted in terms of why (solving the equation as in Q2), while the forms in between are explained by operations (logical steps as in A1). On Q2 sequence of equations, the focused operation refers to the algorithm for solving equations, even though each step in between also involves operations. Thus, the unit of representation that conveys the objective is centered on mathematical representation as a set of sequential forms, rather than on every single..... form.

According to Peirce's semiotics, a sign is not limited to symbols but can be anything. Extending this framework, when we broaden mathematical representation beyond algebraic forms to include figural representations, we consider 2-D diagrams as symbols. Since the signs on semiotics are unusual associated with operations, we retain the word symbol instead of sign. Within this extension, drawing and manipulation can also be regarded as operations (Piaget, 1972). It is illustrated in later Sections in the case of diagrams.

3. Extension to Figural Representation

Here, the theory is illustrated in the case of figural representation in figure 2 (Isoda, 2015).

3-1. A Case Study

Isoda posed the question Q4 to make clear the significance of figural representation to a graduate student who would like to study function of figural representations in algebra.

Following is a summary of discussion:

Firstly, on the board and without using a diagram, she solved the problem by deducing an equation with the length set as x (A1 in Figure 2). She obtained $x = \frac{5}{3}$ as the length, but then tried to erase it. When Isoda asked why, she explained that the fraction seemed strange for this problem. Secondly, she re-deduced the equation and obtained a negative width. At that point, she began to doubt the original problem and reconsidered her deduction (A2). Thirdly, she redefined x as the width, re-deduced the equation, and obtained 14 cm as the width (A3). After confirming her answer, Isoda asked her to solve the problem by diagram. She then sketched the problem situation as shown in Figure 2. When Isoda asked her the significance of her drawing, she replied that it was useful for deducing the equation without overlooking “twice,” “double,” or “two times.” Isoda agreed, telling her that her drawing was a sketch, and discussed that it is usually used for scaffolding by teachers to explain deducing equations for original problem sentences. At the initiation stage of equations, students draw diagram but later they do not feel necessity of drawing because they can deduce without drawing. It is functioned for scaffolding.

Q4. Problem.

There is a rectangle whose length is 3 cm longer than width. A rectangle is drowned by doubling the width of this rectangle and tripling its length. The perimeter of the resulting rectangle is 10 cm longer than twice the perimeter of the original rectangle. Find the length of the perimeter of the original rectangle.

(It was posed by only sentence: from the lecture of Hiroshi Nemoto, Inspector of MOE, Japan)

A1:

length x, width $x-3$

$3x$ $2(x-3)$

$2\big(3x+2(x-3)\big)$

$=2(x-3)+10$

A2:

length x, width $x-3$

$3x$ $2(x-3)$

$2(5x-3)=2(2(2x+3)+10$

A3:

length $x+3$, width x

length $3x+9$, width $2x$

$2(5x+9)=2\big(2(2x+3)\big)+10$

…omitted…

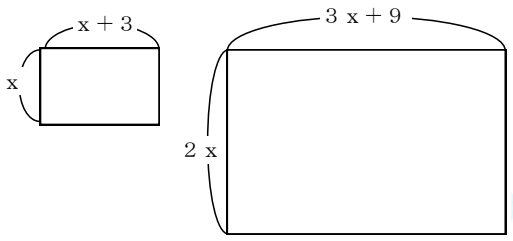


Figure 2. Is the answer, fraction or negative?

After the discussion, Isoda asked her to solve the problem again by drawing diagrams that faithfully represent each sentence of the problem. She did not understand this order and could not proceed. Then, Isoda instructed her to draw the diagram beginning from D1 in Figure 3 for the first sentence ‘There is a rectangle whose length is 3 cm longer than width.’ And asked her to continue drawing D2 for ‘A rectangle is drowned by doubling the width of this rectangle and tripling its length.’ However, she could not go further through interpreting the diagram which she had drawn for the sentence ‘The perimeter of the resulting rectangle is 10 cm longer than twice the perimeter of the original rectangle.’ Isoda asked to confirm which rectangle in the diagram represented the doubling of length and width of the original rectangle in the problem. She marked D3 but did not go further. Then Isoda asked where the 10 cm

measurement appeared in the diagram. By moving the width, she identified the corresponding length and obtained D4.

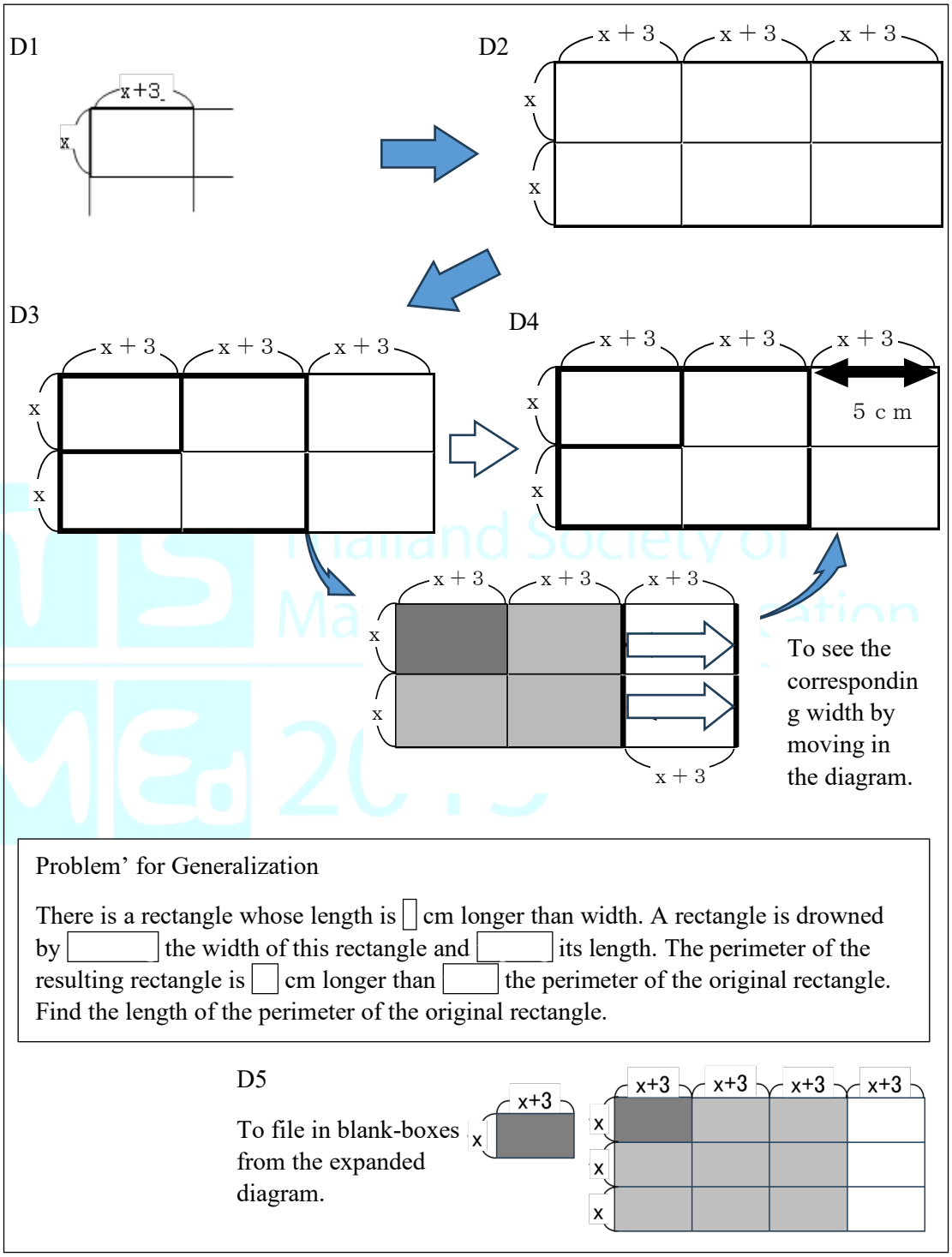


Figure 3. Solution by Diagram

After the session, Isoda reflected from D1 to D4 and asked the further significance of drawing diagrams. She noted that the problem can be solved not only with equations but also with diagrams. And Isoda instructed her that extended problems in Figure 3 can be more easily developed if we understand the structure of this problem through D4 and its correspondences.

In Problem' for Generalization shown in Figure 3, if we insert different numbers into the boxes, we can create new problems. However, we cannot be certain whether these problems are well-posed or ill-posed until we deduce and solve the corresponding equation. If we draw an expanded diagram like D5, using the idea of D4, we can pose new problems whose solutions are already known. In this way, it becomes possible to provide general solutions for this type of problem which developed in the manner of D5 based on unit diagram D1. The solutions by diagrams in Figure 3 provide the local theory for a set of meaningful problems. Even though the algebraic solution by equations is general theory, this figural solution is much significant and stronger on specific situation in local theory.

3-2. Operational Diagram as Mathematical Representation

According to the definition of the unit of mathematical representation (see 2–3), we can distinguish between a sketch such as that in Figure 2, which supports deducing an equation from the problem situation, and the diagrams in Figure 3, which provide a general solution for this problem type (e.g., D5) without deduce equations. The diagrams in Figure 3 form a sequence whose objective is to solve the problem diagrammatically (objective) by arranging the unit rectangle (focused symbol) to generate other rectangles and by shifting widths or lengths to corresponding positions (focused operation). Through these objective, focused symbols, and focused operations, the sequence of diagrams constitutes a unit of mathematical representation.

In contrast, the sketch in Figure 2 is not intended as an object of operation and unnecessary. Indeed, the student reached the correct answer in Figure 2 before drawing, suggesting prior training in equation-based problem solving without diagrams.

On this basis, we define this kind of diagrams that function as mathematical representations as 'operational diagrams.'

Operational diagram has operations/manipulations to change diagram (as symbol): The problem can be solved by the operational diagram. In Figure 3, arrows indicate the operations to change diagram. It represents thinking process step by step. Operational diagrams are distinguished from just a diagram which teachers use only in the purpose of explanation as for scaffolding. Operational diagrams have the rules of drawings which provide the operations for changing diagrams which students implement for specific objective on problem solving.

4. The Mathematical Ideas to See it As

Unit of Mathematical Representations is defined by a sequential representation which is characterized by Objective, Focused Symbols and Operations. Objective provides the focus on interpretation for the ways to see the representation. Mathematical ideas provide the source of objective. To explain how we focus on, here, we confirm the way to see it as is content of teaching in mathematics curriculum.

4-1. Mathematical ideas to fix the ways to see

On Figure 4, there are three ways to find the answer as 27 tiles: Firstly, counting all for which we see it by each single tile, independently. Secondly, sums such as 10 + 9 + 8 for which we see it by distinguished three groups of tiles. And thirdly, multiplying 3×9 for which we move the top tile in the left group to the right group. Here, the tile in Figure 4 becomes manipulative beyond fixed diagram.

Let's confirm it is an operational diagram: For seeing Figure 4 by multiplications, we should develop students who would like to use multiplication with the idea for seeing any number

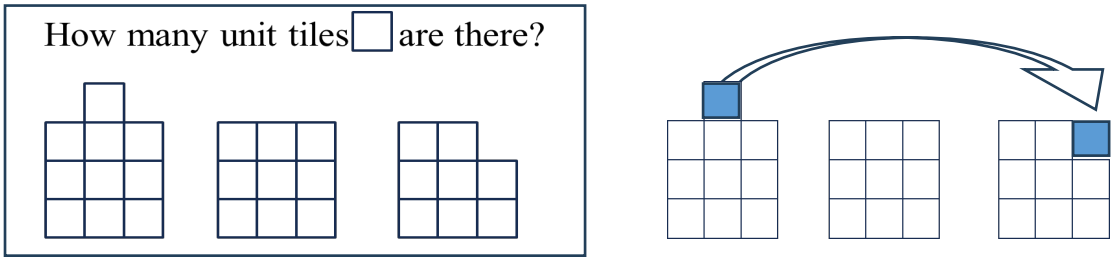


Figure 4. Isoda & Olfos, 2021, pp.124-126

as the new unit for counting (unit for measure). To achieve this idea, it is necessary that students learned the meaning of multiplication which find total amount on number of scaling times by setting a unit of measurement (definition of multiplication: see Isoda&Olfos, 2021) and see 9 as a unit of scaling and would like to apply the raw of 9 on multiplication table which they already memorized. Here, objective of representation is to set the unit for counting to apply multiplication, focused symbols is nine tiles in the middle, and focusing operation is making 9 by moving one tile from left to right.

Isoda and Olfos (2021) demonstrated that students change the ways to see the figure for the same 'how-many' question through learning addition and multiplication from first to third grade in the case of Japan. The discussion implies that we should teach mathematical ideas when we teach mathematical knowledge and skills because the ideas include the ways to see the world for significantly using corresponding knowledge. Due to mathematical representation representing mathematical ideas, from the perspective of mathematical ideas, it tautologically says that mathematical ideas is the source of the objective on mathematical representations which fix the ways to see symbols and operations. On Figure 4, to apply multiplication, students have to have the idea of multiplication to see any number as for the unit for counting.

4-2. Mathematical Thinking Framework in Curriculum to See it As

On Figure 5, the curriculum framework of Southeast Asian Basic Education Standards consisted by I) Value, Attitude, and Habit, II) Ideas, Ways of Thinking and Activities, and III) contents (Isoda, Teh and Gan, 2024). Every component in I)&II) is filled by sample terminology which is replaceable in contextual corresponding, meaningful and understandable words or sentences. The framework illustrates various layers which describe the objectives of mathematics-teaching in every class. Every standard in III) indicates specific content of teaching on curriculum sequence in the subject, here mathematics. Components in I) and II) are general contents of teaching which can be applied to any subjects beyond mathematics as long as replacing words by corresponding words in the subject. For examples, beautifulness as value, pose question as attitude, set as idea and Inductive as thinking, these words can be taught in any subjects. And if we fix the specific content in a subject, it becomes meaningful specifically

in the subject. I) and II) can be learned when appropriate learning process for specific content in III) is given. The appropriate process includes reflection and appreciation. Due to I) and II) are general objectives for any subjects, they can be seen as objectives for human character formation. Mathematics education is the subject to develop human character for I) and II) with the specific content of mathematics in III).

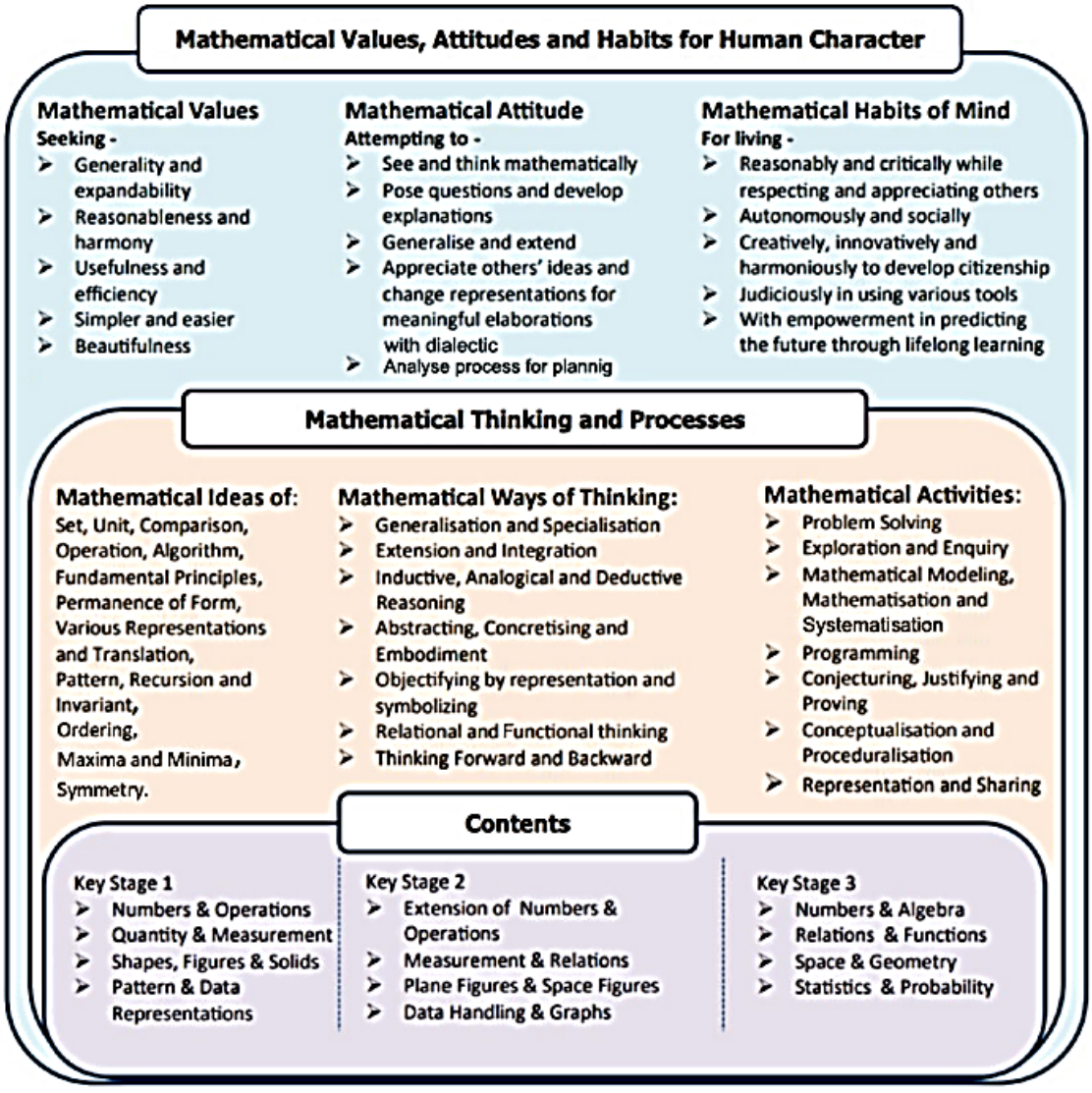


Figure 5. Second Edition of SEA-BES: CCRLS Framework (Isoda, Teh & Gan, 2024., p.8)

Unfortunately, in many cases, teachers only write the objective for specific content in III) and never write I) & II). If we ignore the objective for I) & II), the exercise just for skills in III) looks

like the best way to teach. Even exercise for III, if teachers consider I)&II) as for human character formation, appropriate task sequence for exercise to develop I)&II) is realizable. In today's curriculum, I) & II) are called Generic Competence and III) is called Specific Competence instead of Ability. Due to the term "ability" carries with it a subject-segmented ideology like teachers who only write the objective on III) and inconsiderable I) & II).

On See As Way of Thinking Habit

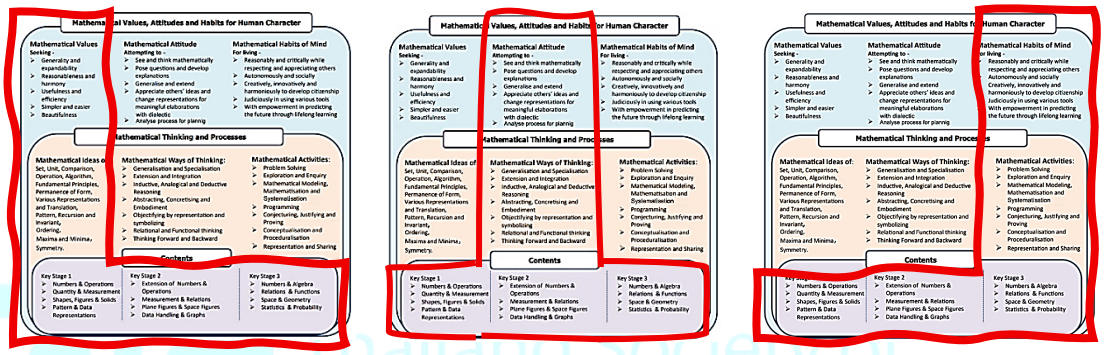


Figure 6. Value, Idea and Content for see it as in the left diagram

Figure 6, from the perspective of See As, both the values in I) and the ideas in II) are used to explain the ways to see. Both are Generic Competence for see it as. In relation to the content as specific competence, there are content specified ideas for see it as explained in 4-1. The content-specific ideas which produce the ways to see only limited functioning within the local

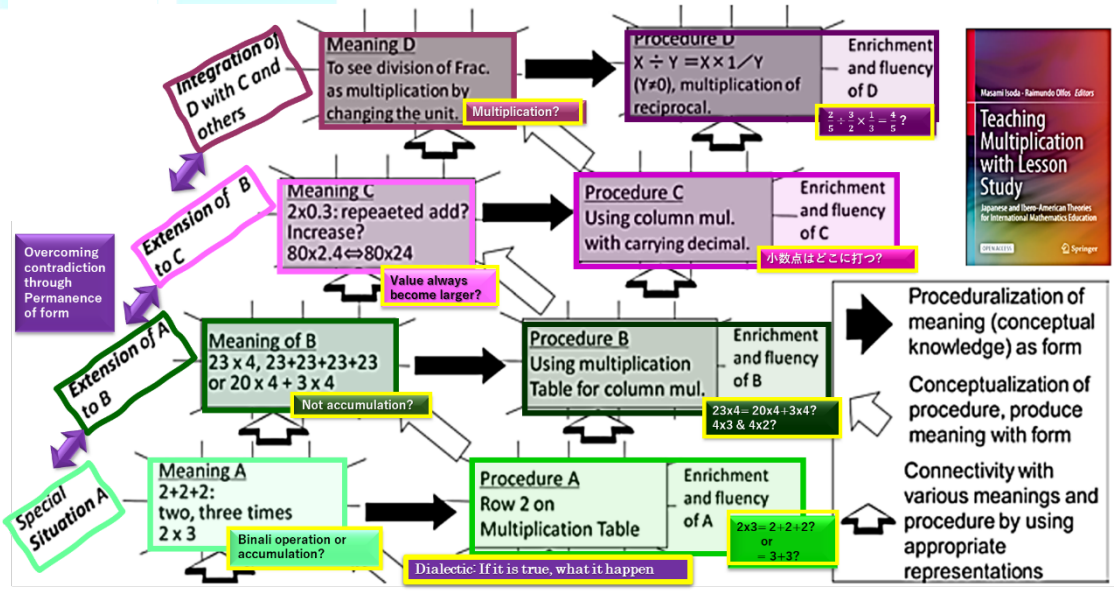


Figure 7. Local theory sequence through extensions (Isoda, Olfos, 2021. p.8,

theory which makes possible to use the ideas. Figure 7 illustrates the curriculum sequence for multiplication which illustrates progressive development. Every box implicates local theory which functions specifically without contradiction but there are contradictions in between next local theories. Sub-box for every local theory illustrates example for contradiction students may confuse and feel strange.

Figure 5 is curriculum framework to explain Standards which might be used to explain objective and teaching sequences for every class. It is terminology to be used for lesson study which explains students' activity and understanding. In traditional terminology to explain cognitive process since 1980s, belief is used for explaining the source for every student's perspective, the way to see, which is emergent in class and observers explain the existence of student's belief (see such as García & Dolores, 2021). On the explanation of ways to see, major difference between Figure 5 curriculum framework and belief in cognitive analysis, Figure 6 is the source of ways to see is explained by past learning what students learned before in curriculum sequence. Instead of cognitive research beginning with observation and discussing the emergence of belief, curriculum provides the objective sequence for teaching. Teachers can expect students' perspective, ways to see, before the class hypothetically based on students already learned or teachers already taught, and teachers try to develop students enabling further ways to see.

5. Adaptation to textbook analysis

Historically, before the invention of algebraic representations, various types of mathematical problems had been solved by figural representations since the ancient era in East and West.

5-1. A Historical Example

In the West, Euclid's Elements (3rd century BC) and Apollonius' Conic Sections (3rd century BC) employed figural representations, which historians call it as "geometric algebra." In the East, the Nine Chapters on the Mathematical Art (Jiǔ Zhāng Suàn Shù, 1st century BC–2nd century AD; Liu Hui edition, 263 AD) and Sun Tzu's Mathematical Classic (Sunzi Suanjing, 5th–6th century AD) categorized problems in terms of equations but solved them using calculation boards.

Here, we analyze the tasks shown in Figure 8 from a Japanese primary school mathematics textbook for Grade 5 (Monbusho, 1939, pp. 71–72), which well reflect the influence of Eastern traditions. Because of printing constraints, the textbook was produced in B6 size (22 cm in diagonal), leaving very limited space for diagrams. Task 1 does not include a diagram, suggesting a solution by equations, but it is also possible to solve using an area diagram. Task 2 is solved through equivalent area transformation. Task 3 is an open task, inviting students to apply the operations on area diagrams.

5-2. Analysis of Task Sequence from the perspective of Mathematical Representations

Here, the task sequence is analyzed from the perspective of objective, focused symbol, and focused operations. In the sequence shown in Figure 8, area diagrams function as operational diagrams. The focused operations are: (i) the area formula of a rectangle, (ii) conservation of area through decomposition and composition, and (iii) area-preserving transformation. Between Task 1 and Task 2,

Thailand Society of Mathematics Education 2013

Task 1

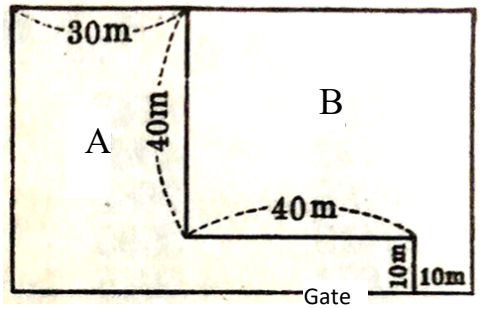
The daily wage for a craftsman is 2.50 yen.

If they work overtime, their daily wage will increase by 0.75 yen for each day.

In February, I took 4 days off and earned 69 yen. How many days of overtime did he work?

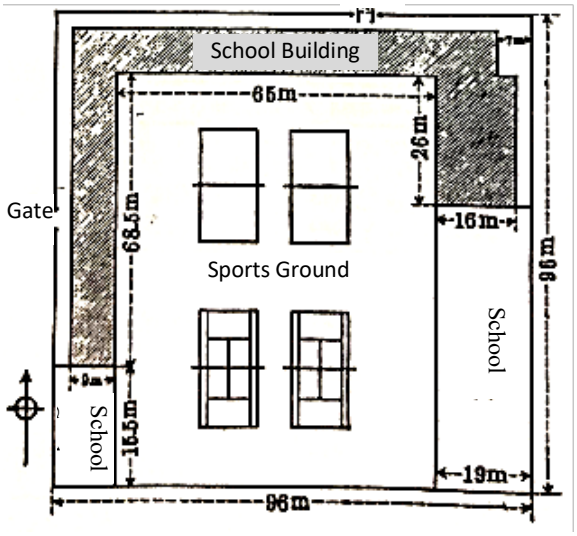
Task 2

The right land is not simple to develop two houses. Thus, to develop the house, they discussed to change the border much simple for easier construction. How shall they change the border?



Task 3.

Right Map is current school environments. Now the city board decided to reform the School Building. How do you plan the new building and how do you plan the construction without closing school.



Algebraic Solution of Task 1

If the overtime days as x,

$$(2.5 + 0.75)x + 2.5((28 - 4) - x) = 69$$

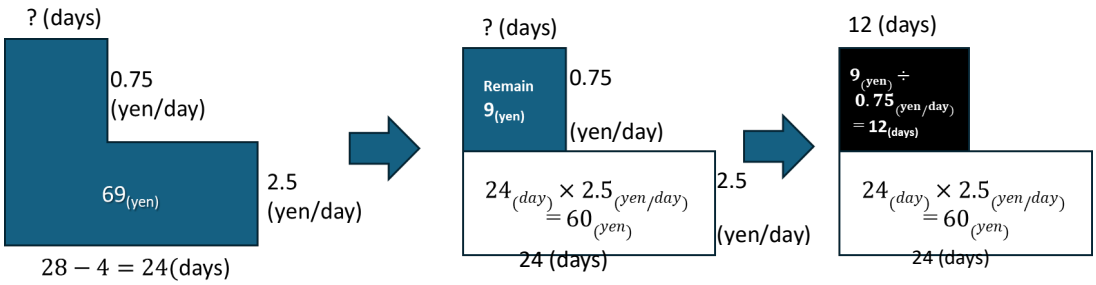
is deduced.

$$3.25x + 2.5 \times 24 - 2.5x = 69$$
$$0.75x = 9$$
$$x = 9 \div 0.75 = 12$$

Figural Solution of Task 1

Ans. 12 days

Figural Solution of Task 1



Figural Solution of Task 2

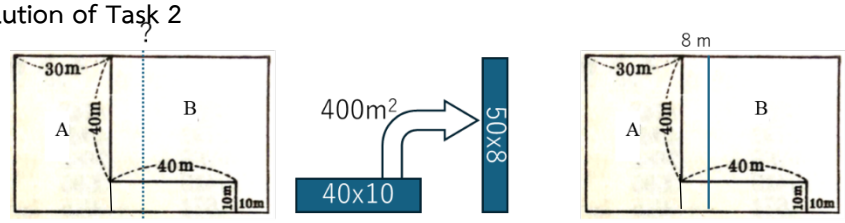


Figure 8. Jinjyo Syougaku Sanjyutu Grade 5 V.2 (Monbusyo, 1940, pp.71-72)

there are two additional problems without diagrams, which extend Task 1 by incorporating ratios.

The L-shaped figure in Figure 8, called ‘gnomon’ in the West and ‘biǎo’ in the Zhōu bì suàn jīng (2nd century BC) in the East, illustrates conservation of area in Task 2. Specifically, it represents the multiplication of length and width (two sides) of a rectangle. Here, the area formula is expressed as the multiplication of the same quantity ‘m’, yielding the area ‘m^2’.

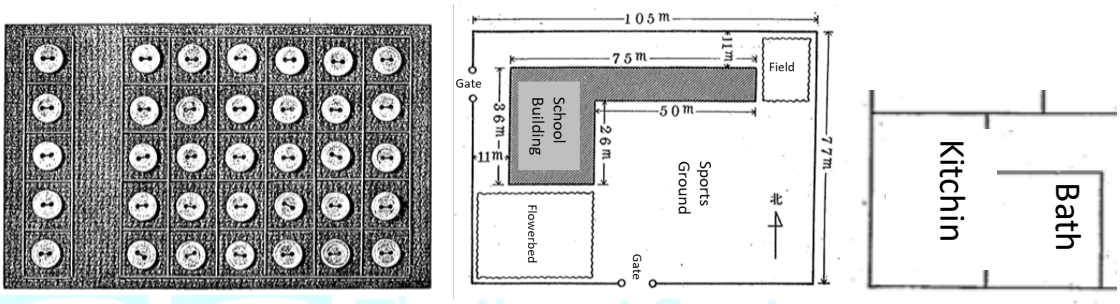


Figure 9. Left: Grade 2 V.1 Multiplication (Monbusyo, 1936, p.50)
Middle: Grade 3 V.2 Parameter (Monbusyo, 1937, p.23)

In contrast, the area diagram in Task 1 does not represent the area. Instead, it illustrates the definition of multiplication in context, ie. $24(day) \times 2.5(yen/day) = 60(yen)$. Drawing such a diagram is more difficult than in Task 2 because it is not a straightforward application of the area formula. Indeed, Task 1 can be solved without a diagram, through hypothetical reasoning such as “if they do not work overtime.” Thus, the objective of Task 1 might be to teach hypothetical reasoning—exploring “what if” and “what if not.” If Task 1 is solved with diagram, a similar problem, which does not have context to use area directly, will be solved by diagrams, before. However, the author could not find similar diagrams to apply multiplication in previous pages but found Figure 9: Especially, multiplication is defined in situation by using multiple such as 6 times of 5 buttons (Figure 9, left).

If the task sequence is taken as given, Task 1 can be revisited after Task 2 with a new objective: introducing the L-shape as an operational diagram (Figure 8) to represent hypothetical reasoning visually. In other words, changing the order—placing Task 2 before Task 1—could

make it easier to extend the area diagram to the definition of multiplication and to reconceptualize the area formula from the perspective of its derivation.

However, it may also be preferable for Task 2 to precede Task 3, since Task 3 asks students to pose new questions based on the conservation of area discussed in Task 2.

6. Theory for Mathematical Representations (Part II)

If discussions about the operational diagram in figure 3 and on the solution of Task 1 in Figure 8 are compared, the solution of Task 2 includes algebraic representation and there is a problem how we read the figure. Thus, the translation between two Mathematical Representations is introduced for interpretation.

6-1. Formalization of Theory

A unit of mathematical representation as a specific set of sequential forms can be defined by objective (Ob), focused symbols (S), and focused operations (Op). A unit is expressed as R(S, Op) with Ob.

In Task 1 (Figure 4), there are three distinct solutions. Equation-based solution: The problem is solved by setting an unknown x and finding its value. The objective is denoted as Obx , the focused symbol as equations Seq, and the focused operation as solving the equation as Opeq. The unit is expressed as Req(Seq,Opeq) with Obx. Hypothetical reasoning: The problem can also be solved without diagrams, through hypothetical reasoning (e.g., “if they do not work overtime”). This is expressed as Rhy(Shy,Ophy) with Obhy. Here, Shy is numbers, Ophy is arithmetic operations under hypothetical reasoning, and Obhy is the hypothetical reasoning to find answer. This arithmetic solution can be written in a diagram, but here, the same reasoning can proceed without it. Figural (diagram-based) solution: A third solution uses an area diagram with arithmetic, interpreting horizontal length as days and vertical width as yen. This is denoted Rdi(Sdi,Opdi) with Obdi. While area is defined as the product of the same quantities, here different quantities are combined (days × yen). Thus, Obdi is “adopting area diagrams for different quantities such as days and yen” to illustrate solution. Sdi is the L-shape, and Opdi is conservation of area for decomposition of the L-shape into two rectangles.



Figure 10. Left: ZHŪ ed. Shìjié, XīnbiǎN shùxué qǐméng (Joseon Dynasty, Copper Type Edition, 1430). https://dc.tulips.tsukuba.ac.jp/s/pub_ja/document/rb_10076713850
Middle: TATEBE Kanehiro, Sangaku keimou genkai taisei (Japan, 1690).
https://kokusho.nijl.ac.jp/biblio/100307114/100?ln=ja

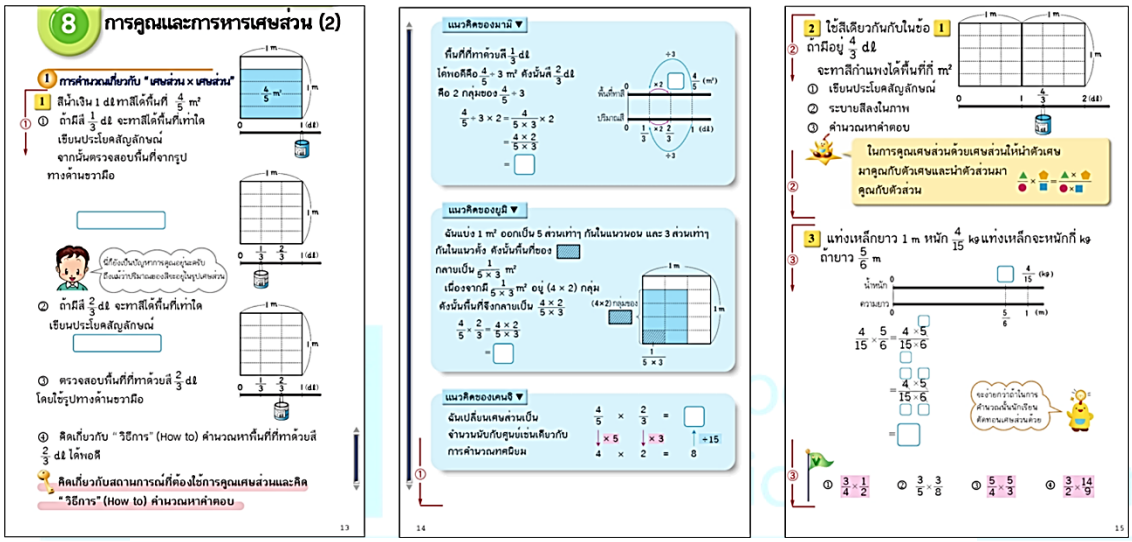
As discussed in Section 4-2, Task 1 can be solved first through Rhy(Shy,Ophy)) with Obhy. After Task 2, students can return to Task 1 and learn Rdi(Sdi,Opdi) with Obdi, if they have not yet encountered figural solutions. This involves a reinterpretation of the L-shape, which originally functioned only as an operational area diagram based on composition or decomposition of rectangles. The original operational diagram, without arithmetic, is denoted RLs(SLs,OpLs)) with ObLs (see Figure 10, right).

Once students learn how to reinterpret the area diagram—treating the horizontal length as days and the vertical width as yen—they can apply Rdi(Sdi,Opdi) with Obdi more generally. For example, in Grade 6 (Monbusho, 1940, Vol. 2, p. 74), the problem states: “There are 20 cranes and tortoises in total. When the total number of legs is 52, find the number of cranes and tortoises.” This can be solved efficiently using a diagram like Rdi(Sdi,Opdi), with the horizontal axis representing cranes and tortoises and the vertical axis representing their legs (2 for cranes, 4 for tortoises). The reasoning “if all are cranes” synchronize with reasoning on Rhy(Shy,Ophy) with Obhy.

Thus, “horizontal as days and vertical as yen” and “horizontal as cranes and tortoises, and vertical as legs” are rules of translation between two mathematical representations: RLs(SLs,OpLs) with ObLs and Rhy(Shy,Ophy) with Obhy. Once this reinterpretation is learned, students can employ Rdi(Sdi,Opdi) with Obdi directly.

The rule of translation—interpreting the horizontal and vertical with different quantities—is denoted by "⇔." Accordingly, the learning sequence is expressed as: $R_{hy}(S_{hy}, Op_{hy})$ in Task 1, $R_{LS}(S_{LS}, Op_{LS})⇔R_{hy}(S_{hy}, Op_{hy})$ after Task 2, and $R_{di}(S_{di}, Op_{di})$ ={$R_{LS}(S_{LS}, Op_{LS})⇔R_{hy}(S_{hy}, Op_{hy})$} in Task 2 and later.

6-2. Historical Emergent



In

Figure 11. Isoda & Maitree (2013) Grade 6, vol.2 pp.13-15

both East and West, the L-shaped figure has been named since ancient times. There were copying constraints for hund writing figure. In the case of the Cranes and Tortoises type problems—which can be represented as Rdi(Sdi,Opdi)—historian Jochi, S. (2015) notes that the first appearance of a diagrammatic solution is found in Tatebe (1690) (Figure 10, middle) on a Chinese textbook originally published during the Yuan Dynasty (Figure 10, left). Prior to Tatebe, the Cranes and Tortoises problem was solved hypothetically, without diagrams, corresponding to Rhy(Shy,Ophy) with Obhy. This historical emergence parallels the learning sequence analyzed on 5-2.

In contemporary Japanese textbooks published by Gakkotosho, this problem type is no longer included. However, RLs(SLs,OpLs) with ObLs which discuss conservation of area is treated (Figure 10, right). Additionally, to explain multiplication and division of fraction, it is used like proportional number line. For example, to explain the division by fraction as the multiplication by the reciprocal number of fraction, area diagrams are used in Figure 11. It is discussed by

the area diagram with besides of the number line, horizontally, to measure the meter. It looks closer to Rdi(Sdi,Opdi) with Obdi style but area diagram is kept the length and width as same quantity instead of changing the quantity to different quantities like Rdi(Sdi,Opdi) with Obdi. In Figure 11, there is another solution by proportional number lines to represent different quantities which is also an operational diagram which provides the solution on diagram. In Figure 11, area diagram is represent the quantity of area and another quantity is given by number line.

7. Final remarks

This paper introduced a part of Theory of Mathematical Representation by defining mathematical representations(R) in terms of objective/context (Ob), focused symbol (S), and focused operation (Op). And illustrated that the objective for focusing is content of teaching in curriculum and adapted textbook analysis.

Within this framework, operational diagrams are distinguished as mathematical representations among various figural representations and confirmed to function as thinking tools. It was illustrated through two examples of operational diagrams, as well as an analysis of textbooks that demonstrated how it can be used to interpret learning sequences. In Japanese origin textbook, a number of operational diagrams are embedded.

From the perspective of Tall (2013), manipulative, concrete material in mathematics learning, is also a part of Embodiment world and it implies that it can be seen as mathematical representation as a part of school mathematics. On this paper, the framework by the objective/context, focused symbol and operation are applicable manipulative representations as a part of mathematical representations which can be distinguished with non-mathematical representations. For example, Origami is an Art by paperfolding, but if the paperfolding is analyzed by geometry and algebraic operations, it becomes the object of mathematical exploration in ORIGAMICS by Haga, K (2008) as a part of mathematical representations.

The theory was originally established to clarifying the activity for mathematization. It is not intended to describe the cognitive processes of individual students, but to serve *kyozaikenkyu*, preparation of teaching materials, especially to analyze the objective of the task in task

sequence in the textbooks and re-establish task sequence to realize it in classroom. In Lesson Study to develop students who think mathematically by and for themselves, task sequence design is a key for kyozaikenkyu because students engage in the learning activity through challenging today's task by using what they already learned.

This paper did not address the ultimate aim to clarify the process of mathematization, Isoda (2015) in Japanese, which described it by transformation of the worlds of mathematical representations. It provides the necessary base to explain the process concrete, semi-concrete and number on TLSOA (add reference of Dr. Maitree and others) in early primary levels. It will be discussed in the next paper.

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